

**The 75th William Lowell Putnam Mathematical Competition**  
**Saturday, December 6, 2014**

A1 Prove that every nonzero coefficient of the Taylor series of  $(1 - x + x^2)e^x$  about  $x = 0$  is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

A2 Let  $A$  be the  $n \times n$  matrix whose entry in the  $i$ -th row and  $j$ -th column is

$$\frac{1}{\min(i, j)}$$

for  $1 \leq i, j \leq n$ . Compute  $\det(A)$ .

A3 Let  $a_0 = 5/2$  and  $a_k = a_{k-1}^2 - 2$  for  $k \geq 1$ . Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

A4 Suppose  $X$  is a random variable that takes on only non-negative integer values, with  $E[X] = 1$ ,  $E[X^2] = 2$ , and  $E[X^3] = 5$ . (Here  $E[Y]$  denotes the expectation of the random variable  $Y$ .) Determine the smallest possible value of the probability of the event  $X = 0$ .

A5 Let  $P_n(x) = 1 + 2x + 3x^2 + \dots + nx^{n-1}$ . Prove that the polynomials  $P_j(x)$  and  $P_k(x)$  are relatively prime for all positive integers  $j$  and  $k$  with  $j \neq k$ .

A6 Let  $n$  be a positive integer. What is the largest  $k$  for which there exist  $n \times n$  matrices  $M_1, \dots, M_k$  and  $N_1, \dots, N_k$  with real entries such that for all  $i$  and  $j$ , the matrix product  $M_i N_j$  has a zero entry somewhere on its diagonal if and only if  $i \neq j$ ?

B1 A *base 10 over-expansion* of a positive integer  $N$  is an expression of the form  $N = d_k 10^k + d_{k-1} 10^{k-1} + \dots + d_0 10^0$  with  $d_k \neq 0$  and  $d_i \in \{0, 1, 2, \dots, 10\}$  for all  $i$ . For instance, the integer  $N = 10$  has two base 10 over-expansions:  $10 = 10 \cdot 10^0$  and the usual base 10 expansion  $10 = 1 \cdot 10^1 + 0 \cdot 10^0$ . Which positive integers have a unique base 10 over-expansion?

B2 Suppose that  $f$  is a function on the interval  $[1, 3]$  such that  $-1 \leq f(x) \leq 1$  for all  $x$  and  $\int_1^3 f(x) dx = 0$ .

How large can  $\int_1^3 \frac{f(x)}{x} dx$  be?

B3 Let  $A$  be an  $m \times n$  matrix with rational entries. Suppose that there are at least  $m + n$  distinct prime numbers among the absolute values of the entries of  $A$ . Show that the rank of  $A$  is at least 2.

B4 Show that for each positive integer  $n$ , all the roots of the polynomial

$$\sum_{k=0}^n 2^{k(n-k)} x^k$$

are real numbers.

B5 In the 75th Annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible  $n \times n$  matrices with entries in the field  $\mathbb{Z}/p\mathbb{Z}$  of integers modulo  $p$ , where  $n$  is a fixed positive integer and  $p$  is a fixed prime number. The rules of the game are:

- (1) A player cannot choose an element that has been chosen by either player on any previous turn.
- (2) A player can only choose an element that commutes with all previously chosen elements.
- (3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy?

B6 Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function for which there exists a constant  $K > 0$  such that  $|f(x) - f(y)| \leq K|x - y|$  for all  $x, y \in [0, 1]$ . Suppose also that for each rational number  $r \in [0, 1]$ , there exist integers  $a$  and  $b$  such that  $f(r) = a + br$ . Prove that there exist finitely many intervals  $I_1, \dots, I_n$  such that  $f$  is a linear function on each  $I_i$  and  $[0, 1] = \bigcup_{i=1}^n I_i$ .